

ARROW TRAJECTORY

In the absence of air resistance the trajectory of an arrow is well known to be a parabola. Introducing aerodynamic drag makes the problem of defining the trajectory in simple analytical functions impossible - the only solution if accurate results are essential is to simulate the dynamics by using the known parameters. This approach is not very useful if we seek to understand how varying the parameters affects the flight path. Luckily, there is an approximate solution which does the trick for most practical purposes - i.e. target and field archery. Clout and popinjay archery requires simulation. The present note covers the two analytical solutions.

No Drag

The only force on an arrow in the absence of drag is due to gravity. This acts downwards with an acceleration of g . Using the conventional Cartesian axes and ignoring z for the present, the equations of motion are:

$$x = ut \text{ and } y = y_0 + vt - \frac{1}{2}gt^2$$

where u and v are the initial velocities in the x and y directions respectively. t represents time and y_0 is the starting height to the arrow. As we are only interested in the trajectory t can be eliminated to give y as a function of x

$$y(x) = y_0 + \frac{v}{u}x - \frac{g}{2u^2}x^2$$

This is the equation of a parabola starting at $x = 0$ and $y = y_0$.

It is convenient to imagine that the reference point of the arrow is the nock since this is the part of the arrow which is actually at zero range initially. If the target were a sheet of paper, the nock would pass through the hole made by the pile so no loss of accuracy for results from this convention.

An example comparing this case with that including drag follows the next section.

Including Drag

To simplify the analysis the following approximations are made:

- All angles are small
- Any wind is blowing in the z direction only

Thus, the forces on the arrow in flight are:

- Drag in a direction opposite to the arrow velocity vector
- Gravity acting only in the y direction with an acceleration $-g$
- Wind pushing the arrow in the z direction with an acceleration $-h$,

The instantaneous drag is given by $-kV^2$ where k is a constant and V is the velocity of the arrow. The drag acceleration acts in the same direction (albeit negatively) as the velocity hence it can be resolved into its Cartesian components with the same direction cosines as the velocity. If we take u , v and w as

the components of the velocity vector in the x , y and z directions respectively, then the drag components in these directions is $-kVu$, $-kVv$ and $-kVw$. The angle between u and V is small so the direction cosine is near unity so we may make the approximation $V = u$.

Acceleration is the rate of change in velocity with respect to time but we are interested in the rate of change with respect to distance in the x direction. In general, $\frac{d}{dt}(\) = \frac{d}{dx}(\) \frac{dx}{dt} = u \frac{d}{dx}(\)$ where the parentheses represent any variable. Using this relationship, the equations of motion are:

$$\begin{aligned} u \frac{du}{dx} &= -ku^2 \text{ or } \frac{du}{dx} = -ku \\ u \frac{dv}{dx} &= -kuv - g \text{ or } \frac{dv}{dx} = -kv - \frac{g}{u} \\ u \frac{dw}{dx} &= -kuw - h \text{ or } \frac{dw}{dx} = -kw - \frac{h}{u} \end{aligned}$$

Note that the form of the equations for the y and z components is the same.

In the x direction the equation may be integrated directly to give $u = u_0 e^{-kx}$ where u_0 is the initial velocity.

In the y direction we note that

$$\begin{aligned} \frac{d}{dx}(ve^{kx}) &= \frac{dv}{dx}e^{kx} + kve^{kx} = \left(\frac{dv}{dx} + kv\right)e^{kx} \\ \text{also, since } u &\text{ is now known } \frac{g}{u} = \frac{g}{u_0} e^{kx} \text{ so from the equations of motion} \\ \frac{d}{dx}(ve^{kx}) &= -\frac{g}{u_0} e^{2kx} \end{aligned}$$

This now integrates to give

$$ve^{kx} = -\frac{g}{2ku_0} e^{2kx} + C$$

where C is a constant i.e.

$$v = Ce^{-kx} - \frac{g}{2ku_0} e^{kx}$$

If $v = v_0$ at $x = 0$ (when $e^{kx} = 1$) then

$$v_0 = C - \frac{g}{2ku_0} \text{ or } C = v_0 + \frac{g}{2ku_0}$$

hence

$$v = v_0 e^{-kx} - \frac{g}{2ku_0} (e^{kx} - e^{-kx})$$

Further integration will yield y as a function of x as follows:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dx}{dt/dt} = \frac{v}{u} = \frac{v_0}{u_0} - \frac{g}{2ku_0^2} (e^{2kx} - 1) \text{ or} \\ y &= \frac{v_0}{u_0} x - \frac{g}{2ku_0^2} \left(\frac{1}{2k} e^{2kx} - x \right) + D \end{aligned}$$

where D is a constant. If $y = y_0$ at $x = 0$ then

$$\begin{aligned} y_0 &= D - \frac{g}{4k^2 u_0^2} \text{ or } D = y_0 + \frac{g}{4k^2 u_0^2} \text{ or} \\ y &= y_0 + \frac{v_0}{u_0} x - \frac{g}{2ku_0^2} \left[\frac{1}{2k} (e^{2kx} - 1) - x \right] \end{aligned}$$

For practical purposes a power series is easier to manipulate to get a feel for what is going on. In the equation for y therefore it is convenient to expand the first few terms of the expression in square brackets.

$$\begin{aligned} [] &= \frac{1}{2k} \left(1 + 2kx + \frac{4k^2 x^2}{2} + \frac{8k^3 x^3}{6} + \dots - 1 \right) - x \\ &= kx^2 \left(1 + \frac{2}{3} kx \right) \end{aligned}$$

Substituting in the equation for y gives

$$y = y_0 + \frac{v_0}{u_0}x - \frac{g}{2u_0^2}x^2 \left(1 + \frac{2}{3}kx\right)$$

Since the form of the equations for the y and z directions is the same, we may write down the z result by inspection.

$$z = z_0 + \frac{w_0}{u_0}x - \frac{h}{2u_0^2}x^2 \left(1 + \frac{2}{3}kx\right)$$

Example

Comparing the results for the trajectory with and without drag it is evident that the only difference is the term $(1 + \frac{2}{3}kx)$. Now from the results quoted in the note on arrow drag, $k = 0.00375 \text{ metres}^{-1}$ so if x is in metres, the term becomes $(1 + 0.0025x)$. Typical values for the other parameters might be

$$y_0 = 1.5 \text{ metres}, u_0 = 50 \text{ metres/sec}, v_0 = 5 \text{ metres/sec}, g = 9.81 \text{ metres/sec/sec}$$

giving

$$y = 1.5 + 0.1x - 0.001962x^2(1 + 0.0025x)$$

